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# ON GENERALIZED FORMULATION OF THE EQUILIBRIUM PROBLEM OF AN ELASTIC STRIP*

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A "nonenergetic" formulation of the boundary value problems of statics of an elastic strip based on the principle of admissible displacements, is studied. The formulation makes possible, in particular, the study of problems concerning the strips of infinite energy, while retaining the external form of the "energetic" formulation /1—3/, and produces unique solvability of the problem under weaker restrictions imposed on the external loads. Such a formulation is also possible for other problems of the theory of elasticity.

1. For an infinite elastic strip $\Omega$ of unit thickness $-\infty < x_1 < \infty,\ 0 \leqslant x_2 \leqslant 1$ with the lower side $x_2 = 0$ rigidly clamped, the principle of admissible displacements has the form

$$\int_\Omega \sigma_{ij}\delta e_{ij}\,dx_1\,dx_2 = \int_\Omega F_i\delta u_i\,dx_1\,dx_2 + \int_\Gamma f_i\delta u_i\,dx_1\,,\quad \sigma_{ij} = E_{ijkl}e_{kl},\quad e_{kl} \equiv e_{kl}(\mathbf{u}) = {}^1/_2(u_{k,l} + u_{l,k}),\quad \delta e_{ij} = e_{ij}(\delta\mathbf{u}) \tag{1.1}$$

where $F_i$ denote the external loads, $f_i$ is the load acting on the top side $\Gamma$ of the strip, $E_{ijkl}$ is the symmetric tensor of elastic constants. It is with respect to these constants as functions of the coordinates $x_1, x_2$ that the boundedness, the Lebesgue measurability and the uniform positive definiteness are assumed on the strip /3/

$$E_{ijkl}\gamma_{kl}\gamma_{ij} \geqslant m\gamma_{ij}\gamma_{ij},\quad m = \mathrm{const} > 0,\quad \gamma_{12} = \gamma_{21}$$

The admissible displacements $\delta\mathbf{u}$ must satisfy, together with the displacements $\mathbf{u}(u_1, u_2)$, the conditions

$$u_i|_{x_2=0} = \delta u_i|_{x_2=0} = 0\quad (i = 1, 2) \tag{1.2}$$

The principle of admissible displacements demands that the equation (1.1) holds for all possible displacements $\delta\mathbf{u}$ satisfying the condition of clamping of the strip. In the energetic formulation of the problem /3/ $\delta\mathbf{u}$ are chosen from the class of functions with "finite energy". This imposes certain specified contraints on the external parameters of the problem, under which the generalized solvability of the problem in the "energetic" class of functions, is easily proved /3/. At the same time the problems in which the load does not decrease at infinity, and in particular where the load is periodic, are excluded from our discussion. In the present paper the class of possible displacements is narrowed, and this leads to the widening of the class of possible loads $F_i$ and $f_i$. We begin by discussing the mathematical apparatus used.

2. The known results due to P. D. Lax /4/ concerning the construction of negative Sobolev spaces, admit an abstract generalization which shall be used below. The corresponding proofs remain almost unchanged, and the terminology is that of /4/.

Let $X$ be a reflexive, complex or real Banach space and $Y$ a normed spaced. Further, let $B(\mathbf{x}, \mathbf{y})$ be a sesquilinear functional continuous in $\mathbf{x} \in X,\ \mathbf{y} \in Y$, i.e. for any complex numbers $\alpha_k, \beta_k$

$$B(\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2, \mathbf{y}) = \alpha_1 B(\mathbf{x}_1, \mathbf{y}) + \alpha_2 B(\mathbf{x}_2, \mathbf{y}),\quad B(\mathbf{x}, \beta_1\mathbf{y}_1 + \beta_2\mathbf{y}_2) = \bar\beta_1 B(\mathbf{x}, \mathbf{y}_1) + \bar\beta_2 B(\mathbf{x}, \mathbf{y}_2)$$

$$|B(\mathbf{x}, \mathbf{y})| \leqslant m_1\|\mathbf{x}\|\ \|\mathbf{y}\|$$

Here and henceforth $m_k$ are positive constants.

Finally, let $\mathbf{y}(\mathbf{x}_1) \in Y$ exist for every element $\mathbf{x}_1 \in X$ and $\mathbf{x}(\mathbf{y}_2) \in X$ for every $\mathbf{y}_2 \in Y$, such that

$$B(\mathbf{x}_1, \mathbf{y}(\mathbf{x}_1)) > 0,\quad B(\mathbf{x}(\mathbf{y}_2), \mathbf{y}_2) > 0,\quad \mathbf{x}_1 \neq 0,\ \mathbf{y}_2 \neq 0$$

According to the above statements every fixed element $\mathbf{y} \in Y$ defines a continuous linear functional on the space $X: f_y(\mathbf{x}) = B(\mathbf{x}, \mathbf{y})$, and its norm is

$$\|f_y\| = \sup_{\|\mathbf{x}\|\leqslant 1}|B(\mathbf{x}, \mathbf{y})| \leqslant m_1\|\mathbf{y}\|$$

We introduce the following negative norm on the space $Y$:

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$$\| y \|_{Y^-} = \sup_{\|x\| \leqslant 1} |B(x, y)|$$

Then $\| f_y \| = \| y \|_{Y^-}$ . We call the completion of the space $Y$ with respect to the negative norm, space $Y^-$

**Theorem 2.1.** The set $X_s'$ of all linear bounded functionals on $X$ can be placed in one-to-one correspondence with the completion $Y^-$ in which the norm is preserved.

**Proof.** The set F of all linear bounded functionals of the form $f(x) = B(x, y)$, $y \in Y$ given on $X$ is dense in $X_s'$. Indeed, the set F is total on $X$, i.e. the fact that $B(x, y) = 0$ for all $y \in Y$ implies that $x = 0$. If F is not dense in $X_s'$, then a functional $T \neq 0$ can be found in $X_{ss}''$ (which is another, strongly conjugated space) such that $T(f_y) = 0$ for all $f_y \in F$. Therefore, by virtue of the reflexivity $X = X_{ss}''$, there exists an element $t \in X$ such that $T(f) = f(t)$ for all $f \in X_s'$. Therefore $T(f_y) = f_y(t) = 0$ and $t = 0$ which is not possible.

The correspondence between $f_y \in F$ and $y \in Y^-$ preserves the norm and is one-to-one.Since F is dense in $X_s'$, the theorem follows.

Analyzing the proof of Theorem 2.1 we arrive at its alternative formulation in which the theorem becomes analogous to the Riesz theorem of representation of a continuous linear functional in a Hilbert space.

**Theorem 2.2.** Every continous linear functional $f(x)$ defined in the space $X$ can be uniquely represented in the form $f(x) = B(x, y_f)$, $y_f \in Y^-$, and $\| f \| = \| y_f \|$.

**Theorem 2.3.** Every continuous linear functional $g(y)$ on the space $Y^-$ can be uniquely represented in the form

$$g(y) = \overline{B(x_g, y)}$$

with help of the element $x_g \in X$.

The proof of the corresponding Lax theorem /4/ is used, in this case, just as in the proof of Theorem 2.1, and is therefore omitted. Moreover Theorem 2.3 is not utilized directly. In what follows, the spaces and functionals are concretized. The space $X(h)$ is a weight energetic space of vector functions $u(u_1, u_2)$ with the norm

$$\| u \|^2_{X(h)} = \int_\Omega \sigma_{ij} e_{ij} h^2(x_1)\, dx_1\, dx_2$$

where $u$ satisfies the condition (1.2), the "weight" $h(x_1) > 0$ is a smooth function and $h(x_1) \to \infty$ as $|x_1| \to \infty$. We also assume that

$$0 < m_2 < h(E(x_1))\, h^{-1}(x_1) < m_3 \tag{2.1}$$

This shows that $h(x_1)$ can increase exponentially at infinity. In (2.1) $E(x_1)$ is the integer part of the number $x_1$. We have the following lemma, which will be proved in Sect.4.

**Lemma 2.1.** The space $X(h)$ is a Hilbert space. The following inequality holds for any vector function $u(u_1, u_2)$:

$$C(u) \equiv \left(\int_\Omega \frac{|u|^p h^r\, dx_1\, dx_2}{(1 + |x_1|)^\gamma}\right)^{1/p} + \left(\int_\Gamma \frac{|u|^p h^r\, dx_1}{(1 + |x_1|)^\gamma}\right)^{1/p} + \left(\int_\Omega h^2 u_{i,j} u_{i,j}\, dx_1\, dx_2\right)^{1/2} \leqslant m(p)\, \| u \|_{X(h)} \tag{2.2}$$

where the constant $m(p)$ is independent of $u$, while the constants $r = p$, $\gamma = 0$ for $p \geqslant 2$ and $r = 2$, $\gamma = 2(2 - p)^{-1} + \varepsilon$ for $1 \leqslant p < 2$ where $\varepsilon > 0$ and arbitrary. For $h(x_1) = 1$ the corresponding lemma is given in /3/.

A set of smooth vector functions vanishing when $|x_1| > R$ where $R$ is a constant associated with each vector function, is a dense set in $X(h)$. Denoting by $Y(h)$ the space $X(1)$ and introducing a bilinear functional

$$B_1(u, v) = \int_\Omega \sigma_{ij}(v)\, e_{ij}(u)\, dx_1\, dx_2$$

we can confirm that all conditions for which Theorem 2.2 is proved, hold for the triad $X(h)$, $Y(h)$, $B_1(u, v)$. We introduce the space $Y^-(h)$ by completing the set of vector functions $v \in Y(h)$ on the norm

$$\| v \|_{Y^-(h)} = \sup_{\|u\|_{X(h)} \leqslant 1} \int_\Omega \sigma_{ij}(v)\, e_{ij}(u)\, dx_1\, dx_2$$

**Lemma 2.2.** Every continuous linear functional can be uniquely represented (according to Theorem 2.2) on the space $X(h)$ in the form

$$\int_{\Omega} \sigma_{ij}(\mathbf{v}^{\circ}) e_{ij}(\mathbf{u})\, dx_1\, dx_2, \quad \mathbf{u} \in X(h)$$

where $\mathbf{v}^{\circ} \in Y^{-}(h)$. The elements of the space $Y^{-}(h)$ are measurable vector functions possessing first generalized derivatives, summable in the square over any finite part of the strip $\Omega$. Indeed, let $\chi(x_1)$ be an infinitely differentiable function with the following properties: $\chi(x_1) = 1$ when $|x_1| \leqslant N$ and $\chi(x_1) = 0$ when $|x_1| \geqslant N+1$ where $N$ is an arbitrary, fixed number. For any element $\mathbf{v} \in Y^{-}(h)$, $\chi\mathbf{v}$ also belongs to $Y^{-}(h)$ and coincides with $\mathbf{v}$ when $|x_1| \leqslant N$.

The closure of the set of elements of the form $\chi\mathbf{u} \in X(h)$ in the norm of $X(h)$ forms a subspace in $X(h)$ which we denote by $X(h, N)$. The functional $B_1(\chi\mathbf{v}, \mathbf{u})$ on $X(h, N)$ is continuous with respect to the variable $\mathbf{u}$ and represents, by virtue of the Korn inequality /3/, a scalar product on $X(h, N)$. Consequently $\chi\mathbf{v} \in X(h, N)$ and the assertion made above, follows.

3. **Definition 3.1.** We call the vector function $\mathbf{u}_0 \in Y^{-}(h)$ satisfying the equation (1.1) for all $\delta\mathbf{u} \in X(h)$, the generalized solution of the problem of equilibrium of the strip. Thus we have incorporated the principle of possible displacements in the basis of the definition of a generalized solution.

**Theorem 3.1.** Let condition (2.1) and

$$\Big(\int_{\Omega} |F_i|^{p-1/p} \Big(\frac{(1+|x_1|)^{\gamma}}{h^{r}(x_1)}\Big)^{1/p-1} dx_1\, dx_2\Big)^{p/p-1} \equiv A_1(\mathbf{F}) < \infty, \quad \Big(\int_{\Gamma} |f_i|^{p-1/p} \Big(\frac{(1+|x_1|)^{\gamma}}{h^{r}(x_1)}\Big)^{1/p-1} dx_1\Big)^{p/p-1} \equiv A_2(\mathbf{f}) < \infty$$

where the relations connecting the constants $r, \gamma$ and $p$ are given in the formulation of Lemma 2.1, both hold. Then the problem of equilibrium of an elastic strip has a unique generalized solution described in Definition 3.1.

**Proof.** The Hölder inequality and Lemma 2.1 together lead to the following inequality:

$$\Big|\int_{\Omega} F_i \delta u_i\, dx_1\, dx_2 + \int_{\Gamma} f_i \delta u_i\, dx_1\Big| \leqslant 2(A_1(\mathbf{F}) + A_2(\mathbf{f}))\, C(\delta\mathbf{u}) \leqslant m_4 \|\delta\mathbf{u}\|_{X(h)}$$

From this it follows that the linear functional appearing in the right-hand side of (1.1) is continuous with respect to the variable $\delta\mathbf{u}$ in the space $X(h)$. According to Lemma 2.2, it has a unique representation of the form

$$\int_{\Omega} \sigma_{ij}(\mathbf{u}_0) e_{ij}(\delta\mathbf{u})\, dx_1\, dx_2, \quad u_0 \in Y^{-}(h)$$

From the form of (1.1) it follows that $\mathbf{u}_0$ is indeed a unique generalized solution of the problem, and this completes the proof of the theorem.

**Notes.** $1^{\circ}$. Problems of equilibrium of a strip under different boundary conditions and of equilibrium of an elastic strip, can be tackled in exactly the same manner.

$2^{\circ}$. The method of constructing a generalized solution given above can also be used for the problems dealing with finite size regions. Depending on the choice of the space $X$, we can arrive at the concept of a solution belonging to one or another class of generalized functions with certain regularity properties. In particular, we can include the case of nonintegrable loads with various type singularities which disappear from the "energetic" formulation of the problems.

$3^{\circ}$. The conditions of Theorem 3.1 admit the exponential growth of the loads at infinity.

$4^{\circ}$. The construction of a generalized solution given above can be used for general, elliptic, positive definite boundary value problems.

4. **Proof of Lemma 2.1.** The fact that the space $X(h)$ is a Hilbert space follows directly from the form of the norm of $X(h)$ and the Korn inequality /3/. The proof of the estimate (2.2) is identical for the first two integrals appearing in the left-hand side of (2.2), therefore we shall prove only one.

We divide the strip $\Omega$ into squares $K_n$ of unit side. On $K_n$ $n \leqslant x_1 < n+1$. By virtue of the boundary conditions and the Korn inequality /3/ the following inequality holds for the vector function and for all $1 \leqslant p < \infty$:

$$\Big(\int_{K_n} |\mathbf{u}|^{p}\, dx_1\, dx_2\Big)^{2/p} \leqslant m_5 \int_{K_n} \sigma_{ij}(\mathbf{u}) e_{ij}(\mathbf{u})\, dx_1\, dx_2$$

We multiply both sides of the above expression by $h^2(n)$ and carry out summation over $n$. This yields

$$\sum_{n=-\infty}^{\infty}\left(\int_{K_n} h^p(n)\,|\mathbf{u}|^p\,dx_1\,dx_2\right)^q \leqslant m_5 \sum_{n=-\infty}^{\infty}\int_{K_n} h^2(n)\,\sigma_{ij}\varepsilon_{ij}\,dx_1\,dx_2\,, \quad (q = 2p^{-1}) \tag{4.1}$$

and the latter gives at once the estimate (2.2) for the first integral, with (2.1) taken into account and $p = 2$. The estimate for the third integral of (2.2) is obtained in exactly the same manner.

Let now $1 \leqslant p < 2$. To obtain the estimate (2.2) it is sufficient to define the weight function $\rho(x_1) = (1 + |x_1|)^\gamma$ so that the following inequality holds:

$$\left(\sum_{n=-\infty}^{\infty}\int_{K_n}\rho(n)\,h^p(n)\,|\mathbf{u}|^p\,dx_1\,dx_2\right)^q \leqslant m_6 \sum_{n=-\infty}^{\infty}\left(\int_{K_n} h^p(n)\,|\mathbf{u}|^p\,dx_1\,dx_2\right)^q$$

or

$$\left(\sum_{n=-\infty}^{\infty}\rho_{(n)}a_n\right)^q \leqslant m_6 \sum a_n{}^q, \quad a_n = \int_{K_n} h^p(n)\,|\mathbf{u}|^p\,dx_1\,dx_2$$

which is the same. Since the sequences $\{a_n\}$ and $\{a_{-n}\}$, $0 \leqslant n < \infty$ are arbitrary and belong to the space $l^q$ of sequences summable to degreee $q$, it follows that the sequences $\{\rho(\pm n)\}$ must belong to the conjugate space $l^{2/(2-p)}$. This is possible if the index $\gamma$ in the expression for the function $\rho(x_1)$ $\gamma > 2(2-p)^{-1}$. Taking into account (2.1), we now obtain the estimate (2.2) for $1 \leqslant p < 2$.

Let now $p > 2$. By virtue of the Jensen inequality

$$\left(\sum |a_n|^q\right)^{1/q} \geqslant \sum |a_n|, \quad q < 1$$

we obtain, taking into account (2.1), the inequality

$$\sum_{n=-\infty}^{\infty}\left(\int_{K_n} h^p(n)\,|u|^p\,dx_1\,dx_2\right)^q \geqslant \left(m_2\int_{\Omega} h^p(x_1)\,|u|^p\,dx_1\,dx_2\right)^q$$

which, combined with (4.1), completes the proof of the estimate (2.2).

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## REFERENCES

1. EIDUS D.M., Contact problem of the theory of elasticity. Matem. sb., Vol.54, No.3, 1954.

2. FIKERA G. Theorems of Existence in the Theory of Elasticity. Moscow, "Mir", 1974.

3. VOROVICH I.I., ALEKSANDROV V.M. and BABESHKO V.A., Nonclassical Mixed Problems of the Theory of elasticity. Moscow, "Nauka", 1974.

4. YOSIDA K. Functional Analysis. Springer-Verlag, Berlin, Heidelberg, New York, 1974.

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